# Characterization of Lipschitz continuous DC functions*

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**Abstract**

We give a necessary and sufficient condition for a difference of convex (DC, for short) functions, defined on a locally convex space, to be Lipschitz continuous. Our criterion relies on the intersections of the $\varepsilon$-subdifferentials of the involved functions.

**Key words.** DC functions, Lipschitz continuity, Integration formulas, $\varepsilon$-subdifferential

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## 1 Introduction

In this paper, we work with a (Hausdorff) real locally convex topological vector space $X$ whose dual is denoted by $X^*$. The duality product is denoted by $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$, and the zero vector (in $X$ and $X^*$) by $\theta$.

Classical integration formulas ([8, 9]) have been first established in the Banach spaces setting for proper lower semicontinuous (lsc, for short) convex functions using the Fenchel subdifferential, which is defined for a given function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x$ in the domain of $f$, $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$, by

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \text{for all } y \in X\}.$$

These results have been extended outside the Banach space ([1, 7]) and the nonconvex settings ([3]) by using the $\varepsilon$-subdifferential mapping, defined for $\varepsilon > 0$

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by

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid f(y) - f(x) \geq \langle y - x, x^* \rangle - \varepsilon \text{ for all } y \in X\}.$$

In this paper we exploit an idea, recently used in [6], to establish several characterizations for the Lipschitz character of the difference of convex (DC, for short) functions. As a consequence, if the Lipschitz constant is equal to 0 then we obtain an integration formula guaranteeing the coincidence of the involved functions up to an additive constant. The main result is presented in Theorem 1 in a slightly more general form, valid in the locally convex spaces setting, which characterizes the domination of the variations of DC functions by means of a convex continuous functions. The desired integration formula is obtained in Theorem 5.

# 2 The main result

The desired results providing the characterization of Lipschitz DC functions will be given in Theorem 5, which is a consequence of the following theorem.

In what follows, $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are two given functions with a common domain

$$D := f^{-1}(\mathbb{R}) = g^{-1}(\mathbb{R}),$$

assumed nonempty and convex.

**Theorem 1** *Let $h : X \longrightarrow \mathbb{R}$ be a continuous convex function such that $h(\theta) = 0$. Then, the following statements are equivalent*:

(i) *$f$ and $g$ are convex, lsc on $D$, and satisfy*

$$f(x) - g(x) \leq f(y) - g(y) + h(x - y) \quad \textit{for all } x, y \in D.$$

(ii) *For each $x \in D$*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + \partial_\varepsilon h(\theta) \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + \partial_\varepsilon h(\theta) \quad \textit{for all } \varepsilon \in (0, \delta).$$

(iv) *For each $x \in D$*

$$\partial_\varepsilon f(x) \cap (\partial_\varepsilon g(x) + \partial_\varepsilon h(\theta)) \neq \emptyset \quad \textit{for all } \varepsilon > 0.$$

(v) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\partial_\varepsilon f(x) \cap (\partial_\varepsilon g(x) + \partial_\varepsilon h(\theta)) \neq \emptyset \quad \textit{for all } \varepsilon \in (0, \delta).$$

**Proof.** (i) $\Longrightarrow$ (ii). Since $f$ is proper ($\operatorname{dom} f \neq \emptyset$), convex and lsc on $D$, for any given $\varepsilon > 0$ the $\varepsilon$-subdifferential operator $\partial_\varepsilon f$ is nonempty on $D$ ([11, Prop. 2.4.4(iii)]). For $x \in D$, we define the function $\widetilde{g} : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\widetilde{g} := g + f(x) - g(x)$$

so that by (i) the inequality $f \leq \widetilde{g} + h(\cdot - x)$ holds, as well as $f(x) = \widetilde{g}(x) + h(\theta) = \widetilde{g}(x)$. Notice that $\operatorname{cl} \widetilde{g} = \operatorname{cl} g + f(x) - g(x)$, where cl refers to the corresponding lsc envelope. Hence, as $g$ is lsc on $D$, $\operatorname{cl} \widetilde{g}$ coincides with $g + f(x) - g(x)$ on $D$, which implies that it is proper. Therefore, since ([4, Lemma 15])

$$\operatorname{cl}(\widetilde{g} + h(\cdot - x)) = \operatorname{cl} \widetilde{g} + h(\cdot - x) = \operatorname{cl} g + h(\cdot - x) + f(x) - g(x)$$

and $\partial_\delta(\operatorname{cl} \widetilde{g})(x) = \partial_\delta \widetilde{g}(x) = \partial_\delta g(x)$ (for all $\delta > 0$), by appealing to the sum rule of the $\varepsilon$-subdifferential (e.g., [11, Theorem 2.8.3]) we get

$$\begin{aligned} \partial_\varepsilon f(x) &\subset \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \left( \partial_{\varepsilon_1} (\operatorname{cl} \widetilde{g})(x) + \partial_{\varepsilon_2} h(\theta) \right) \\ &= \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \left( \partial_{\varepsilon_1} g(x) + \partial_{\varepsilon_2} h(\theta) \right) \subset \partial_\varepsilon g(x) + \partial_\varepsilon h(\theta); \end{aligned}$$

showing that (ii) holds.

The implication (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) and (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) are obvious.

(v) $\Longrightarrow$ (i). We fix $x, y \in D$ and take an arbitrary number $\varepsilon > 0$. For $m = 1, 2, \cdots$ we denote

$$x_{m,i} := x + \frac{i}{m}(y - x) \quad \text{for } i = 0, 1, \cdots, m.$$

Then, by the current assumption (v) for each $i$ and $m$ there exists $\gamma_{m,i} \in (0, m^{-1})$ such that

$$\partial_{m^{-1}\gamma\varepsilon} f(x_{m,i}) \cap \left[ \partial_{m^{-1}\gamma\varepsilon} g(x_{m,i}) + \partial_{m^{-1}\gamma\varepsilon} h(\theta) \right] \neq \emptyset \quad \text{for all } \gamma \in (0, \gamma_{m,i}).$$

Set

$$\gamma_m := \min_{i \in \{1, \cdots, m\}} \gamma_{m,i},$$

so that $\gamma_m > 0$, and choose $u^*_{m,i} \in \partial_{m^{-1}\gamma_m\varepsilon} f(x_{m,i})$, $v^*_{m,i} \in \partial_{m^{-1}\gamma_m\varepsilon} g(x_{m,i})$ and $w^*_{m,i} \in \partial_{m^{-1}\gamma\varepsilon} h(\theta)$ such that $u^*_{m,i} = v^*_{m,i} + w^*_{m,i}$ for $i = 1, ..., m-1$. In this way, if $u^* \in \partial_\varepsilon f(x)$ and $v^* \in \partial_\varepsilon g(y)$ are given we write

$$\begin{aligned} f(x_{m,1}) - f(x) &\geq \frac{1}{m} \langle y - x, u^* \rangle - \varepsilon \\ f(x_{m,i+1}) - f(x_{m,i}) &\geq \frac{1}{m} \left\langle y - x, u^*_{m,i} \right\rangle - m^{-1}\gamma_m \varepsilon \qquad (i = 1, ..., m-1) \\ g(x_{m,i-1}) - g(x_{m,i}) &\geq -\frac{1}{m} \left\langle y - x, v^*_{m,i} \right\rangle - m^{-1}\gamma_m \varepsilon \qquad (i = 1, ..., m-1) \\ g(x_{m,m-1}) - g(y) &\geq -\frac{1}{m} \langle y - x, v^* \rangle - \varepsilon. \end{aligned}$$

Adding up these inequalities and using the facts that $x_{m,m} = y$ and $x_{m,0} = x$, together with $u^*_{m,i} = v^*_{m,i} + w^*_{m,i}$, we obtain that

$$f(y) - f(x) + g(x) - g(y) \geq \frac{1}{m}\langle y - x, u^* - v^*\rangle + \frac{1}{m}\sum_{i=1}^{m-1}\langle y - x, w^*_{m,i}\rangle - 2(m-1)m^{-1}\gamma_m\varepsilon - 2\varepsilon.$$

Thus, since $w^*_{m,i} \in \partial_{m^{-1}\gamma\varepsilon}h(\theta)$ we deduce that

$$f(y) - f(x) + g(x) - g(y) \geq \frac{1}{m}\langle y - x, u^* - v^*\rangle - \frac{m-1}{m}h(x-y) - 2(m-1)m^{-1}\gamma_m\varepsilon - 2\varepsilon$$

which gives us, as $m$ goes to $\infty$ (recall that $0 < \gamma_m \leq m^{-1}$),

$$f(y) - f(x) + g(x) - g(y) \geq -h(x-y) - 2\varepsilon.$$

Hence, by letting $\varepsilon$ go to 0 we get

$$f(x) - g(x) \leq f(y) - g(y) + h(x-y);$$

that is, (i) follows. ■

The particular case $h := 0$ in Theorem 1 yields a new integration result, which relies on the intersection of the $\varepsilon$-subdifferentials of the nominal functions. We will denote by $f_D$ and $g_D$ the restrictions of $f$ and $g$ to $D$, respectively.

**Corollary 2** (cf. [2, Corollary 2.5]) *The following statements are equivalent*:

(i) *$f$ and $g$ are convex, lsc on $D$, and $f_D - g_D$ is constant.*

(ii) *For each $x \in D$*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \quad \text{for all } \varepsilon > 0.$$

(iii) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \quad \text{for all } \varepsilon \in (0, \delta).$$

(iv) *For each $x \in D$*

$$\partial_\varepsilon f(x) \cap \partial_\varepsilon g(x) \neq \emptyset \quad \text{for all } \varepsilon > 0.$$

(v) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\partial_\varepsilon f(x) \cap \partial_\varepsilon g(x) \neq \emptyset \quad \text{for all } \varepsilon \in (0, \delta).$$

The following corollary, giving a criterion for integrating the Fenchel subdifferential, is an immediate consequence of Corollary 2 in view of the straightforward relationships $\partial f(x) \subset \partial_\varepsilon f(x)$ and $\partial g(x) \subset \partial_\varepsilon g(x)$ for every $x \in D$ and every $\varepsilon > 0$.

**Corollary 3** (cf. [6, Theorem 1]) *The following statements are equivalent*:

(i) *For each* $x \in D$
$$\emptyset \neq \partial f(x) \subset \partial g(x).$$

(ii) *For each* $x \in D$
$$\partial f(x) \cap \partial g(x) \neq \emptyset.$$

(iii) *For each* $x \in D$
$$\emptyset \neq \partial f(x) = \partial g(x).$$

*If these statements hold, then* $f$ *and* $g$ *are convex, lsc on* $D$, *and* $f_D - g_D$ *is constant.*

**Remark 4** a) The preceding results remain true if $X$ is an arbitrary locally convex real topological vector space, not necessarily Hausdorff. Indeed, the equivalence between the convex and lsc character of a function and the nonemptiness of its $\varepsilon$-subdifferentials is a reformulation of the Fenchel-Moreau Theorem, the validity of which in non-Hausdorff spaces has been proved by S. Simons [10, Theorem 10.1].

b) The equivalence between (i) and (ii) in Corollary 2 also follows from a well-known characterization of global minima of DC functions due to J.-B. Hiriart-Urruty [5, Theorem 4.4]. Indeed, according to this characterization, if $f$ and $g$ are convex then one has $\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x)$ for all $\varepsilon > 0$ if and only if $x$ is a global minimum of $f_D - g_D$. Hence, that condition holds for every $x \in D$ if and only if every $x \in D$ is a global minimum of $f_D - g_D$, which is obviously equivalent to $f_D - g_D$ being constant on $D$.

From now on we suppose that $X$ is a normed space with a norm denoted by $\|\cdot\|$ whose the dual norm is $\|\cdot\|_*$. We use $B_*(\theta, K)$ to denote the closed ball in $(X^*, \|\cdot\|_*)$ with center $\theta$ and radius $K \geq 0$, and for $A, B \subset X^*$ we set
$$d(A,B) := \inf\{\|a-b\|_* : a \in A,\ b \in B\},$$
with the convention that $d(A,B) := +\infty$ if $A$ or $B$ is empty.

At this moment, we easily get the main result of the paper by taking $h := K\|\cdot\|$ in Theorem 1:

**Theorem 5** *Let* $K \geq 0$. *Then, the following statements are equivalent*:

(i) $f$ *and* $g$ *are convex, lsc on* $D$, *and* $f_D - g_D$ *is Lipschitz with constant* $K$.

(ii) *For each* $x \in D$
$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + B_*(\theta, K) \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each* $x \in D$ *there exists* $\delta > 0$ *such that*
$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + B_*(\theta, K) \quad \textit{for all } \varepsilon \in (0, \delta).$$

(iv) *For each* $x \in D$

$$\partial_{\varepsilon} f(x) \cap [\partial_{\varepsilon} g(x) + B_{*}(\theta, K)] \neq \emptyset \quad \textit{for all } \varepsilon > 0.$$

(v) *For each* $x \in D$ *there exists* $\delta > 0$ *such that*

$$\partial_{\varepsilon} f(x) \cap [\partial_{\varepsilon} g(x) + B_{*}(\theta, K)] \neq \emptyset \quad \textit{for all } \varepsilon \in (0, \delta).$$

(vi) *For each* $x \in D$

$$d(\partial_{\varepsilon} f(x), \partial_{\varepsilon} g(x)) \leq K \quad \textit{for all } \varepsilon > 0.$$

(vii) *For each* $x \in D$ *there exists* $\delta > 0$ *such that*

$$d(\partial_{\varepsilon} f(x), \partial_{\varepsilon} g(x)) \leq K \quad \textit{for all } \varepsilon \in (0, \delta).$$

**Proof.** The proofs of the equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) follow from Theorem 1 by observing that $\partial_{\varepsilon}(K\|\cdot\|)(\theta) = B_{*}(\theta, K)$. The implications (iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) are obvious. To prove (vii) $\Longrightarrow$ (i), given $x \in D$ we notice that (vii) implies the existence of $\delta > 0$ such that, for all $\gamma > 0$,

$$\partial_{\varepsilon} f(x) \cap [\partial_{\varepsilon} g(x) + B_{*}(\theta, K + \gamma)] \neq \emptyset \quad \text{for all } \varepsilon \in (0, \delta).$$

Hence, by the equivalence between (v) and (i), $f$ and $g$ are convex, lsc on $D$, and $f_D - g_D$ is Lipschitz with constant $K + \gamma$. Therefore, since $\gamma$ is arbitrary, $f_D - g_D$ is Lipschitz with constant $K$. ■

Observing that statements (i), (iv), (v), (vi) and (vii) in Theorem 5 are symmetric in $f$ and $g$, it turns out that, under the assumptions of this theorem, statements (ii) and (iii) are also symmetric; therefore, if one has

$$\emptyset \neq \partial_{\varepsilon} f(x) \subset \partial_{\varepsilon} g(x) + B^{*}(\theta, K) \quad \text{for all } \varepsilon > 0$$

for each $x \in D$, then one also has

$$\emptyset \neq \partial_{\varepsilon} g(x) \subset \partial_{\varepsilon} f(x) + B^{*}(\theta, K) \quad \text{for all } \varepsilon > 0$$

for each $x \in D$. We thus obtain the following corollary:

**Corollary 6** *Let* $K \geq 0$. *If some (hence all) of the statements* (i)–(vii) *of Theorem* 5 *holds, then for every* $x \in D$ *and every* $\varepsilon > 0$ *the Hausdorff distance between* $\partial_{\varepsilon} f(x)$ *and* $\partial_{\varepsilon} g(x)$ *does not exceed the constant* $K$.

**Corollary 7** *The following statements are equivalent*:
(i) $f$ *and* $g$ *are convex, lsc on* $D$, *and* $f_D - g_D$ *is constant.*
(ii) *For each* $x \in D$

$$d(\partial_{\varepsilon} f(x), \partial_{\varepsilon} g(x)) = 0 \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each* $x \in D$ *there exists* $\delta > 0$ *such that*

$$d(\partial_{\varepsilon} f(x), \partial_{\varepsilon} g(x)) = 0 \quad \textit{for all } \varepsilon \in (0, \delta).$$

From the previous result we obtain a complement to Corollary 3:

**Corollary 8** *The following statements are equivalent*:
(i) *For each* $x \in D$
$$\emptyset \neq \partial f(x) = \partial g(x).$$
(ii) *For each* $x \in D$
$$d(\partial f(x), \partial g(x)) = 0.$$

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